QUASI-PROJECTIVE BRAUER CHARACTERS

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Abstract We study p-Brauer characters of a finite group G which are restrictions of generalized characters vanishing on p-singular elements for a fixed prime p dividing the order of G. Such Brauer characters are called quasi-projective. We show that for each irreducible Brauer character there exists a minimal p-power, say $p^{a(\varphi)}$ such that $p^{a(\varphi)}\varphi$ is quasi-projective. The exponent $a(\varphi)$ only depends on the Cartan matrix of the block to which φ belongs. Moreover $p^{a(\varphi)}$ is bounded by the vertex of the module afforded by φ, and equality holds in case that G is p-solvable. We give some evidence for the conjecture that $a(\varphi) = 0$ occurs if and only if φ belongs to a block of defect 0. Finally, we study indecomposable quasi-projective Brauer characters of a block B. This set is finite and corresponds to a minimal Hilbert basis of the rational cone defined by the Cartan matrix of B.

Keywords: block, defect, Cartan matrix, Brauer character, quasi-projective character, projective module

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1. Introduction

Throughout this paper let p always denote a prime and let G be a finite group. By $\mathrm{Irr}(G)$ resp. $\mathrm{Irr}(B)$ we denote the set of ordinary irreducible characters of G resp. of a p-block B, and by $\mathrm{IBr}_p(G)$ resp. $\mathrm{IBr}_p(B)$ that of irreducible p-Brauer characters with respect to a p-modular splitting system. We put $l(B) = |\,\mathrm{IBr}_p(B)|$. Finally, we write Φ_φ for the ordinary character associated to the projective cover of the module corresponding to $\varphi \in \mathrm{IBr}_p(G)$. If χ is a generalized ordinary character of G, then χ° denotes the restriction of χ on the set of p-regular elements.

Quasi-projective ordinary characters Λ, i.e., ordinary characters Λ of the form $\Lambda = \sum_{\varphi\in \mathrm{IBr}_p(G)} a_\varphi \Phi_\varphi$ with $a_\varphi \in \mathbb{Z}$ have been studied in [15]. Here we consider the analogue for p-Brauer characters where we often suppress the underlying prime p.

Definition 1.1. A Brauer character Φ of G is called *quasi-projective* if

$$\Phi = \sum_{\varphi \in \mathrm{IBr}_p(G)} a_\varphi \Phi_\varphi^\circ \text{ with } a_\varphi \in \mathbb{Z}.$$

We call Φ *projective* if $a_\varphi \geq 0$ for all $\varphi \in \mathrm{IBr}_p(G)$, i.e., ϕ is the Brauer character of a projective module.

Note that $\sum_{\varphi \in \mathrm{IBr}_p(G)} a_\varphi \Phi_\varphi$ with $a_\varphi \in \mathbb{Z}$ is a generalized ordinary character which vanishes on p-singular elements and vice versa (see [11, Theorem 2.13 and Corollary 2.17]).

Due to Dickson's Theorem [11, Corollary 2.14] the p-part $|G|_p$ of the order of G divides the degree of any quasi-projective Brauer character.

In order to study arbitrary quasi-projective Brauer characters the following definition is useful.

Definition 1.2. A quasi-projective Brauer character Φ is called *indecomposable quasi-projective* if it can not be written as

$$\Phi = \Phi_1 + \Phi_2$$

where the Φ_i are quasi-projective and not zero.

Observe that an indecomposable quasi-projective Brauer character always belongs to a block (see for instance [4, Ch. IV, Lemma 3.14]).

Whenever we need an explicit Cartan matrix of a particular block B we use the information given in [10]. The computation of $a(\varphi)$ for $\varphi \in \mathrm{IBr}_p(B)$ is mostly carried out with the software package 4ti2 (see [6]).

We would like to mention here that there is some overlapping with results of an unpublished paper of Zeng [16] written in Chinese.

2. Quasi-projective Brauer characters

Let B be a p-block of G with Cartan matrix C and let $\Phi = \sum_{\varphi \in \mathrm{IBr}_p(B)} a_\varphi \Phi_\varphi^\circ$ with $a_\varphi \in \mathbb{Z}$ be a quasi-projective Brauer character of B. If we put $a = (a_\varphi)_{\varphi \in B}$ as a column vector and use the symmetry of the Cartan matrix, we get

$$\begin{aligned} \Phi &= \textstyle\sum_{\varphi \in \mathrm{IBr}_p(B)} a_\varphi \Phi_\varphi^\circ \\ &= \textstyle\sum_{\varphi \in \mathrm{IBr}_p(B)} a_\varphi \sum_{\psi \in \mathrm{IBr}_p(B)} c_{\varphi,\psi} \psi \\ &= \textstyle\sum_{\psi \in \mathrm{IBr}_p(B)} (\sum_{\varphi \in \mathrm{IBr}_p(B)} c_{\psi,\varphi} a_\varphi) \psi, \end{aligned}$$

where $(\sum_{\varphi \in \mathrm{IBr}(B)} c_{\psi,\varphi} a_\varphi)$ is the entry in Ca at position ψ. Thus Φ is a quasi-projective Brauer character if and only if $Ca \geq 0$.

Theorem 2.1. *Let B be a p-block of G of defect d.*

a) *For each* $\varphi \in \mathrm{IBr}_p(B)$ *there is a minimal p-power, say* $p^{a(\varphi)}$ *with* $0 \le a(\varphi) \le d$ *such that* $p^{a(\varphi)}\varphi$ *is a quasi-projective Brauer character. Obviously,* $p^{a(\varphi)}\varphi$ *is an indecomposable quasi-projective Brauer character.*
b) *If* $n\varphi$ *is quasi-projective for* $\varphi \in \mathrm{IBr}_p(B)$ *for some* $n \in \mathbb{N}$*, then* $p^{a(\varphi)} \mid n$*.*
c) *For all* $\varphi \in \mathrm{IBr}_p(B)$ *of height zero we have* $a(\varphi) = d$*.*
d) *If* $p^{d_1} \le \ldots \le p^{d_{l-1}} < p^{d_l} = p^d$ *are the elementary divisors of the Cartan matrix* C *of* B*, then there exists a labbeling of the irreducible Brauer characters of* B *such that* $d_i \le a(\varphi_i)$ *for all* $i = 1, \ldots, l$*.*

Proof. a) Let e_φ denote the column vector which has a 1 at position φ and zeros elsewhere. Solving $Ca = e_\varphi$ via Cramer's rule we see that each entry a_ψ in a is a quotient $\pm\frac{b}{\det C}$, where b is the determinant of an $(l-1)\times(l-1)$ submatrix of C. Since $\det C$ is a power of p [4, Ch. IV, Theorem 3.9], we may choose a minimal $a(\varphi) \in \mathbb{N}_0$ such that $p^{a(\varphi)}a \in \mathbb{Z}^l$. In particular $Cp^{a(\varphi)}a = p^{a(\varphi)}e_\varphi$. This shows that $p^{a(\varphi)}\varphi$ is a quasi-projective Brauer character.

To see that $a(\varphi) \le d$ for all $\varphi \in \mathrm{IBr}_p(B)$ we argue as follows. Let $p^{d_1}, p^{d_2}, \ldots, p^{d_l} = p^d$ be the elementary divisors of C where $d_1 \le \ldots \le d_{l-1} < d_l = d$ (for the last inequality see [4, Ch. IV, Theorem 4.16]). Since $\prod_{i=1}^{l-1} p^{d_i}$ divides the determinant of every $(l-1)\times(l-1)$ submatrix of C, we get $\frac{b}{\det C} \in \frac{1}{p^d}\mathbb{Z}$. This shows that $a(\varphi) \le d$ for all $\varphi \in \mathrm{IBr}(B)$.
b) The assertion follows immediately from the proof of part a).
c) If $\varphi \in \mathrm{IBr}_p(B)$ is of height zero, then $\varphi(1)_p = \frac{|G|_p}{p^d}$. Since $p^{a(\varphi)}\varphi$ is quasi-projective by part a), we obtain

$$|G|_p \mid p^{a(\varphi)}\varphi(1)_p = \frac{p^{a(\varphi)}|G|_p}{p^d}.$$

It follows that $d \le a(\varphi)$ and we are done since $a(\varphi) \le d$, by part a).
d) According to a), there exists an integer matrix A such that

$$CA = \begin{pmatrix} p^{a(\varphi_1)} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & p^{a(\varphi_l)} \end{pmatrix}$$

is diagonal. By the definition of elementary divisors there are matrices P and Q over the integers of determinant ± 1 such that

$$C = P \begin{pmatrix} p^{d_1} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & p^{d_l} \end{pmatrix} Q.$$

Thus

$$QA = \begin{pmatrix} p^{-d_1} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & p^{-d_l} \end{pmatrix} P^{-1} \begin{pmatrix} p^{a(\varphi_1)} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & p^{a(\varphi_l)} \end{pmatrix}.$$

Note that P^{-1} is a matrix over $\mathbb{Z}$. Therefore we may write $P^{-1} = (x_{ij})$ with $x_{ij} \in \mathbb{Z}$ and we get

$$QA = \begin{pmatrix} x_{11}p^{a(\varphi_1)-d_1} & x_{12}p^{a(\varphi_2)-d_1} & \dots & x_{1l}p^{a(\varphi_l)-d_1} \\ x_{21}p^{a(\varphi_1)-d_2} & x_{22}p^{a(\varphi_2)-d_2} & \dots & x_{2l}p^{a(\varphi_l)-d_2} \\ \vdots & \vdots & \vdots & \vdots \\ x_{l1}p^{a(\varphi_1)-d_l} & x_{l2}p^{a(\varphi_2)-d_l} & \dots & x_{ll}p^{a(\varphi_l)-d_l} \end{pmatrix}.$$

Since $p \nmid \det P^{-1}$ there exists a sequence $(1, i_1), (2, i_2), \dots, (l, i_l)$ where the i_j form a set of size l such that

$$p \nmid x_{1i_1}x_{2i_2} \cdots x_{li_l}.$$

This implies $a(\varphi_{i_1}) \geq d_1$, $a(\varphi_{i_2}) \geq d_2$, ..., $a(\varphi_{i_l}) \geq d_l$ and the proof is complete. □

Definition 2.2. For $\varphi \in \mathrm{IBr}_p(G)$ we call $p^{a(\varphi)}$ the *Hilbert divisor* of φ.

The name Hilbert divisor has been chosen since $p^{a(\varphi)}\varphi$ corresponds naturally to an element in the (unique) minimal Hilbert basis defined by the cone of the Cartan matrix of the block to which φ belongs (see Section 4). As the proof of Theorem 2.1 a) shows we may find for any Brauer character β in B a minimal p-power $p^{a(\beta)}$such that $p^{a(\beta)}\beta$ is quasi-projective. However, $p^{a(\beta)}\beta$ corresponds natuarally to an element of the Hilbert basis if and only if $p^{a(\beta)}\beta$ is indecomposable quasi-projective (which is in general hard to test).

As an immediate consequence of Theorem 2.1 we get a strengthend version of a Theorem of Brauer for irreducible Brauer characters (see [4, Ch. IV, Theorem 1.2]).

Corollary 2.3. *If $\varphi \in \mathrm{IBr}_p(B)$, then $\tilde{\varphi}$ defined by*

$$\tilde{\varphi}(x) = \begin{cases} p^{a(\varphi)}\varphi(x), & \textit{if } x \textit{ is a } p\textit{'-element,} \\ 0, & \textit{otherwise} \end{cases}$$

is a generalized character of B.

Remark 2.4. One can not do better in Corollary 2.3. If we choose $n(\varphi) \in \mathbb{N}_0$ minimal such that

$$\tilde{\varphi}(x) = \begin{cases} p^{n(\varphi)}\varphi(x), & \text{if } x \text{ is a } p\text{'-element,} \\ 0, & \text{otherwise} \end{cases}$$

is a generalized character, then $\tilde{\varphi}(x) = \sum_{\psi \in \mathrm{IBr}_p(B)} a_\psi \Phi_\psi$ with complex numbers a_ψ, by [11, Theorem 2.13]. However, the coefficients a_ψ are even integers since

$a_\psi = (\tilde{\varphi}, \psi)^\circ = (\tilde{\varphi}, \psi)$ and ψ is an integer linear combination of ordinary irreducible characters restricted to p-regular elements (see [11, Corollary 2.16]). This shows that $n(\varphi) = a(\varphi)$.

Corollary 2.5. *If $C^{-1} = (c^{\varphi\psi})$ is the inverse of the Cartan matrix C, then $c^{\varphi\varphi} = (\varphi, \varphi)^\circ > 0$ for all $\varphi \in \mathrm{IBr}_p(G)$.*

Proof. We may assume that C is the Cartan matrix of a block B. By Theorem 2.1, we have $\sum_{\psi \in \mathrm{IBr}_p(B)} a_\psi \Phi_\psi = p^{a(\varphi)}\varphi$, hence $Ca = p^{a(\varphi)} e_\varphi$. Since C is positive definite, we obtain

$$0 < a^T C a = p^{a(\varphi)} a_\varphi,$$

hence $a_\varphi > 0$. The entry a_φ in a is equal to the entry at position φ in $a = C^{-1}Ca = p^{a(\varphi)} C^{-1} e_\varphi$ which is $p^{a(\varphi)} c^{\varphi\varphi}$, hence $c^{\varphi\varphi} > 0$. Finally, note that $c^{\varphi\varphi} = (\varphi, \varphi)^\circ$, by [4, Chap. IV, Lemma 3.7]. □.

Examples 2.6. a) Let $G = A_5$ be the alternating group on 5 letters and let B_0 be the principal 2-block of G. The elementary divisors of the Cartan matrix of B_0 are $4, 1, 1$. The Hilbert divisors $2^{a(\varphi)}$ for $\varphi \in \mathrm{IBr}_2(B_0)$ are $4, 2, 2$.
b) Let $G = A_4$ be the alternating group on 4 letters and let B_0 be the principal 2-block of G (which is the full group algebra). Note that the defect group of B_0 is a Klein four group as in a). The elementary divisors of the Cartan matrix are again $4, 1, 1$, but the Hilbert divisors $2^{a(\varphi)}$ for $\varphi \in \mathrm{IBr}_2(B_0)$ are now $4, 4, 4$.

For $\varphi \in \mathrm{IBr}_p(G)$ we denote by $vx(\varphi)$ a vertex of the module in characteristic p afforded by φ.

Proposition 2.7. *For $\varphi \in \mathrm{IBr}_p(G)$ we have $p^{a(\varphi)} \leq |vx(\varphi)|$.*

Proof. According to Remark 2.4 it is sufficient to show that

$$\tilde{\varphi}(x) = \begin{cases} |vx(\varphi)|\varphi(x), & \text{for } x \text{ a } p\text{'-element,} \\ 0, & \text{otherwise} \end{cases}$$

is a generalized character. By a result of Brauer ([4], Chap.IV, Theorem 1.1), we may prove that $\tilde{\varphi}|_E$ is a generalized character for any elementary subgroup E of G. Let M be the module afforded by φ and let $V = vx(M) = vx(\varphi)$. Clearly, $M|_E = \oplus_{i=1}^s M_i$ with indecomposable E-modules M_i. We denote by φ_i the Brauer character of M_i and by V_i its vertex $vx(M_i) \leq V$. Since E is an elementary group we get by Green's indecomposable theorem [8, Chap. VII, Theorem 16.2] that $M_i = N_i^E$, where N_i is an indecomposable module for the group $V_i \times Q$. Let ψ_i be the Brauer character of N_i. Thus, if $v_i = |V_i|$ and $v = |V|$, then $\tilde{\varphi}|_E = \sum_{i=1}^s p^{v-v_i} \tilde{\psi}_i^E$ where

$$\tilde{\psi}_i(x) = \begin{cases} v_i \psi_i(x), & \text{for } x \text{ a } p\text{'-element,} \\ 0, & \text{otherwise} \end{cases}$$

Note that ψ_i is the sum of irreducible Brauer characters of $V_i \times Q$, say $\psi_{i,j}$ for $j = 1, \ldots, n_i$. Since an irreducible Brauer character of $V_i \times Q$ is of height zero, we get $v_i = p^{a(\psi_{i,j})}$ for all j, by Theorem 2.1 c). Therfore $\tilde{\psi}_i$ is a generalized character according to Corollary 2.3. It follows that $\tilde{\varphi}|_E$ is a generalized character and we are done. □

Proposition 2.8. *Let G be p-solvable. If $\varphi \in \mathrm{IBr}_p(G)$, then $p^{a(\varphi)} = |vx(\varphi)|$. In particular, $\Phi_\varphi(1) = p^{a(\varphi)}\varphi(1)$.*

Proof. By Proposition 2.7, we have $p^{a(\varphi)} \leq |vx(\varphi)|$. Furthermore, $|G|_p \mid p^{a(\varphi)}\varphi(1)$ since $p^{a(\varphi)}\varphi(1)$ is quasi-projective. The p-solvability of G implies $\varphi(1) \mid \frac{|G|_p}{|vx(\varphi)|}$, hence

$$|G|_p \mid \frac{p^{a(\varphi)}|G|_p}{|vx(\varphi)|}$$

which implies $|vx(\varphi)| \leq p^{a(\varphi)}$ and we are done. □

We would like to mention that the assertion of Proposition 2.8 does not hold in general as Example 2.6 a) already shows. Note that the vertices of all simple modules in the principal 2.block of A_5 have order 4, but for the Hilbert divisors we only get $4, 2, 2$.

Furthermore, the condition $p^{a(\varphi)} = |vx(\varphi)|$ for all $\varphi \in \mathrm{IBr}_p(G)$ does not imply in general that G is p-solvable. As an example the group $7^2 : 2.L_2)7).2$ mod 7 may serve.

Lemma 2.9. *If B is a p-block of a p-solvable group with $l(B) > 1$, then $c_{\varphi\varphi} < p^{a(\varphi)}$.*

Proof. According to Proposition 2.8 we have $\Phi_\varphi(1) = p^{a(\varphi)}\varphi(1)$. The condition $|\mathrm{IBr}_p(B)| > 1$ implies that there exists $\varphi \neq \psi \in \mathrm{IBr}_p(B)$ with $c_{\varphi\psi} \neq 0$. Since $c_{\varphi\varphi} \leq |vx(\varphi)|$ according to [5], the degree of Φ_φ forces $c_{\varphi\varphi} < |vx(\varphi)| = p^{a(\varphi)}$. □

Proposition 2.10. *If B is a p-block with a cyclic defect group of order p^d, then $a(\varphi) = d$ for all $\varphi \in \mathrm{IBr}_p(B)$.*

Proof. First we consider the case that the Brauer tree is a star with e edges and exceptional vertex in the center with multiplicity $\frac{p^d-1}{e}$. In this case the Cartan matrix C of B is of the form $C = (\mu + \delta_{\varphi\psi})_{\varphi,\psi\in\mathrm{IBr}_p(B)}$ where $\mu = \frac{p^d-1}{e}$ and $e = l(B)$. In particular, $\det C = e\mu + 1 = p^d$. If $e = 1$, the assertion is clear, by Theorem 2.1. If $e > 1$, then for $\varphi \neq \psi \in \mathrm{IBr}_p(B)$ the entry at (φ, ψ) in C^{-1} equals $\pm\frac{\mu}{e\mu+1} = \pm\frac{\mu}{p^d}$ where μ and p^d are coprime, and we are done again.

To deal with the general case let D be a defect group of B, $D_1 \leq D$ of order p and $N_1 = N_G(D_1)$. According to [1, section 14, Theorem 3], there exists a block b_1 of N_1 with defect group D such that $B = b_1^G$. Morover, the correspondence between the irreducible modules V of b_1 and $U$of B is given by

$$V^G = U \oplus P, \quad U|_{N_1} = V \oplus W, \tag{1}$$

where P is a projective G-module and W is a direct sum of a projective N_1-module and modules lying in blocks different from b_1. By [1, section 14, Theorem 2], the Brauer tree of b_1 is a star with e edges and exceptional vertex in the center with multiplicity $\frac{p^d-1}{e}$. Thus, according to the first part of the proof, we have $a(\varphi) = d$ for all $\varphi \in \mathrm{IBr}_p(b_1)$. If $\psi \in \mathrm{IBr}_p(B)$ corresponds to $\varphi \in \mathrm{IBr}_p(b_1)$, then by (1) we have

$$\psi|_{N_1} = \varphi + \Psi + \Lambda \tag{2}$$

where Ψ is the character of a projective module and all constituents of Λ belong to blocks different from b_1. Note that $p^{a(\psi)}\psi|_{N_1}$ is quasi-projective and $a(\psi) \leq d$. By (2), it follows that $p^{a(\psi)}\varphi$ is a quasi-projective Brauer character of b_1, hence $a(\psi) = a(\varphi) = d$. □

Remark 2.11. According to Proposition 2.8 we have $p^{a(\varphi)} = |vx(\varphi)|$ for all $\varphi \in \mathrm{IBr}_p(G)$ provided G is p-solvable. Unfortunately, the converse does not hold true. For instance, we may take a simple non-abelian group with a cyclic Sylow p-subgroup. Then, by a result of Erdmann [3], the vertices of all $\varphi \in \mathrm{IBr}_p(B)$ coincide with the defect group of the block B. On the other hand, by Proposition 2.10, we have $a(\varphi) = d$ if d is the defect of the block to which φ belongs.

Proposition 2.12. *Let B be a p-block of G of defect d. Furthermore, suppose that $\varphi \in \mathrm{IBr}_p(B)$ with height $\mathrm{ht}(\varphi)$. Then the following are equivalent.*

a) $a(\varphi) = \mathrm{ht}(\varphi) = 0$.
b) $d = 0$*, i.e., B is of defect* 0.

Proof. Assume that a) holds true. According to Theorem 2.1, the assumption $\mathrm{ht}(\varphi) = 0$ implies $a(\varphi) = d$. Thus $d = 0$, by assumption. The converse is obviously true. □

Remark 2.13. Let $\varphi \in \mathrm{IBr}_p(B)$ where B is a p-block with defect group D and defect d. Suppose that $a(\varphi) = 0$ and $\mathrm{ht}(\varphi) \neq 0$. Then we have

a) $\mathrm{ht}(\varphi) \geq 2$.
b) If $\mathrm{ht}(\varphi) = 2$, then $d = 2$, $D \cong C_p \times C_p$ and p odd.

Proof. Let $|G|_p = p^a$. Since $p^a \mid \varphi(1)_p = p^{a-d+\mathrm{ht}(\varphi)}$ we have $\mathrm{ht}(\varphi) \geq d$. If $\mathrm{ht}(\varphi) = 1$, then, by Proposition 2.12, we have $d = 1$. Thus D is cyclic of order p and $a(\varphi) = d = 1$, by Proposition 2.10.
b) As in a) we have $d \neq 0$. If $d = 1$, then $p^{a+1} \mid \varphi(1)$, contradicting the Theorem in [12]. Thus $d = 2$. Note that D can not be cyclic. Furthermore $p = 2$ does not occur, since by [2], B is Morita equivalent to KD, KA_4 or $B_0(KA_5)$ where K is an appropriate field of characteristic 2.

□

Due to Proposition 2.8, Proposition 2.10 and many other examples we may state the following conjecture.

Conjecture 2.14. (**Hilbert divisor Conjecture**) Let B be a p-block of G. If $a(\varphi) = 0$ for some $\varphi \in \mathrm{IBr}_p(B)$, then B is a block of defect 0. In other words: If $\varphi \in \mathrm{IBr}_p(G)$ is quasi-projective, then φ is the character of an irreducible projective module.

Proposition 2.15. *Let $C^{-1} = (c^{\beta\gamma})$ denote the inverse of the Cartan matrix of a p-block B. For $\varphi \in \mathrm{IBr}_p(B)$ the following are equivalent.*

a) $a(\varphi) = 0$.
b) $c^{\varphi\lambda} = (\varphi, \lambda)^\circ \in \mathbb{Z}$ *for all* $\lambda \in \mathrm{IBr}_p(B)$.

Proof. If $\sum_{\psi \in \mathrm{IBr}_p(B)} a_\psi \Phi_\psi = \varphi$ with $\varphi \in \mathrm{IBr}_p(G)$, then

$$\mathbb{Z} \ni a_\lambda = (\sum_{\psi \in \mathrm{IBr}_p(B)} a_\psi \Phi_\psi, \lambda)^\circ = (\varphi, \lambda)^\circ = c^{\varphi\lambda}$$

for all $\lambda \in \mathrm{IBr}_p(B)$. Conversely, suppose that $c^{\varphi\lambda} \in \mathbb{Z}$ for all $\lambda \in \mathrm{IBr}_p(B)$. By Theorem 2.1, we have $\sum_{\psi \in \mathrm{IBr}_p(B)} a_\psi \Phi_\psi^\circ = p^{a(\varphi)}\varphi$ with $a_\psi \in \mathbb{Z}$. It follows

$$a_\lambda = (\sum_{\psi \in \mathrm{IBr}_p(B)} a_\psi \Phi_\psi, \lambda)^\circ = (p^{a(\varphi)}\varphi, \lambda)^\circ = p^{a(\varphi)} c^{\varphi\lambda}.$$

Thus $p^{a(\varphi)} \mid a_\lambda$ for all $\lambda \in \mathrm{IBr}_p(B)$ and we get $\sum_{\psi \in \mathrm{IBr}_p(B)} \frac{a_\psi}{p^{a(\varphi)}} \Phi_\psi^\circ = \varphi$. □

Remark 2.16. Let B be a p-block of G and suppose that $a(\varphi) = 0$ for some $\varphi \in \mathrm{IBr}_p(B)$. According to Corollary 2.3 the map $\tilde{\varphi}$ defined by

$$\tilde{\varphi}(x) = \begin{cases} \varphi(x), & \text{for } x \text{ a } p\text{'-element}, \\ 0, & \text{otherwise} \end{cases}$$

is a generalized character of B. The Conjecture 2.14 says that $\tilde{\varphi}$ is even a character.

Recall that the exponent $\exp(G)$ of a finite p-group G (p a prime) is the maximal order of an element in G.

Examples 2.17. a) The principal 2-block B_0 of the alternating group A_6 has three irreducible Brauer characters of degree $1, 4, 4$. The vertex of the 4-dimensional characters are Klein four-groups, hence of exponent 2. The Hilbert divisors $2^{a(\varphi)}$ for $\varphi \in \mathrm{IBr}_2(B_0)$ are $8, 2, 2$.
b) The principal 2-block B_0 of the simple Mathieu group M_{12} contains three irreducible Brauer characters of degree $1, 10, 44$. All of them have a Sylow 2-subgroup as vertex (see [13]). Thus $\exp(vx(\varphi)) = 2^3$ for all $\varphi \in \mathrm{IBr}_2(B_0)$. One computes $2^6, 2^5, 2^4$ for the Hilbert divisors $2^{a(\varphi)}$.

c) Let $G = Sz(8)$ be the smallest Suzuki group and let $p = 2$. For the inverse of the Cartan matrix C of the principal 2-block we obtain

$$C^{-1} = \begin{pmatrix} \frac{55}{64} & -\frac{9}{16} & -\frac{9}{16} & -\frac{9}{16} & -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \\ -\frac{9}{16} & \frac{7}{4} & -\frac{1}{4} & -\frac{1}{4} & -1 & 1 & 0 \\ -\frac{9}{16} & -\frac{1}{4} & \frac{7}{4} & -\frac{1}{4} & 1 & 0 & -1 \\ -\frac{9}{16} & -\frac{1}{4} & -\frac{1}{4} & \frac{7}{4} & 0 & -1 & 1 \\ -\frac{1}{4} & -1 & 1 & 0 & 2 & 0 & 0 \\ -\frac{1}{4} & 1 & 0 & -1 & 0 & 2 & 0 \\ -\frac{1}{4} & 0 & -1 & 1 & 0 & 0 & 2 \end{pmatrix}$$

This shows that for the Hilbert divisors $2^{a(\varphi)}$ we have $2^6, 2^4, 2^4, 2^4, 2^2, 2^2, 2^2$. According to a well-known result of Dipper [7, Section 8.9, Theorem] the Sylow 2-subgroup S of G is the vertex of any $\varphi \in \mathrm{IBr}_2(B_0)$. Furthermore $\exp(S) = 4$.

Based on many further examples, on blocks with cyclic defect groups or blocks of p-solvable groups one is tempted to ask the following.

Question 2.18. Let B be a p-block and let $\varphi \in \mathrm{IBr}_p(B)$. Do we always have

$$p^{a(\varphi)} \geq \exp(vx(\varphi))?$$

Furthermore, the following problem might also be of interest.

Question 2.19. What can we say if $a(\varphi) = d$ for all $\varphi \in \mathrm{IBr}_p(B)$ where d denotes the defect of B? It happens, for instance, if the defect group is cyclic (see Proposition 2.10).
Warning: It does not imply that the defect group of B is abelian as the principal 2-block of $PSL(2, 7)$ shows (see Example 4.1 b)), which belongs to the class $\mathrm{D}(3\mathcal{K})$ in Erdmann's classification of tame blocks. In this class we have $a(\varphi) = d$ for all irreducible Brauer characters in the block.

Remark 2.20. In contrast to ordinary irreducible characters the condition $|G|_p \mid \varphi(1)$ for $\varphi \in \mathrm{IBr}_p(G)$ does not imply that φ belongs to a p-block of defect 0. As an example we may take $G = McL$ and $\varphi \in \mathrm{IBr}_2(B_0)$ of degree $2^9 \cdot 7$ (see [4, Ch. IV, Some open problems, page 166]), where B_0 denotes the prinical block. Note that $|G|_2 = 2^7$. A direct computation shows that $a(\varphi) = 2$. Furthermore we get $7, 6, 6, 5, 5, 5, 5$ for $a(\psi)$ where $\psi \in \mathrm{IBr}_2(B_0)$ and $\psi \neq \varphi$..

3. Quasi-projective Brauer characters and normal subgroups

By [4, Ch. III, Corollary 4.13] we know that a normal p-subgroup N of G is contained in the vertex of any irreducible Brauer character φ of G. Moreover

$|vx(\varphi)| = |N||vx(\bar{\varphi})|$ where $\bar{\varphi}$ is the Brauer character of the module afforded by φ but regarded as a module for $\bar{G} = G/N$. Note that N is contained in the kernel of φ.

Proposition 3.1. *Let N be a normal p-subgroup of G with $|N| = p^n$, $|G|_p = p^a$ and $\bar{G} = G/N$. If $\varphi \in \mathrm{IBr}_p(G)$, then*

$$a(\bar{\varphi}) + n - \mathrm{ht}(\bar{\varphi}) \leq a(\varphi).$$

Proof. Let B be the block of defect d to which φ belongs and let $\bar{B}$ be the block of defect $\bar{d}$ to which $\bar{\varphi}$ belongs. Note that

$$p^{a-d+\mathrm{ht}(\varphi)} = \varphi(1)_p = \bar{\varphi}(1)_p = p^{\bar{a}-\bar{d}+\mathrm{ht}(\bar{\varphi})}. \tag{3}$$

By [11, Theorem 9.9 (a)], we have

$$d - \bar{d} = n + m \tag{4}$$

where $m \geq 0$. Thus equation (3) implies that

$$\mathrm{ht}(\varphi) = \mathrm{ht}(\bar{\varphi}) + m. \tag{5}$$

Since $p^{a(\varphi)}\varphi$ is quasi-projective we get

$$p^a \mid p^{a(\varphi)}\varphi(1)_p = p^{a(\varphi)+a-d+\mathrm{ht}(\varphi)},$$

hence

$$a(\varphi) \geq d - \mathrm{ht}(\varphi).$$

By (4) and (5), it follows

$$a(\varphi) \geq \bar{d} + n + m - (\mathrm{ht}(\bar{\varphi}) + m) \geq a(\bar{\varphi}) + n - \mathrm{ht}(\bar{\varphi}).$$

□

Example 3.2. The group $G = N : A_6$ where $N \lhd G$ and elementary abelian of order 2^4 is a maximal subgroup of M_{22}. We denote by $\bar{G}$ the factor group $G/N \cong A_6$. For the prime $p = 2$ the Cartan matrix of G is

$$C = \begin{pmatrix} 80 & 48 & 44 & 12 & 12 \\ 48 & 30 & 26 & 7 & 7 \\ 44 & 26 & 27 & 8 & 8 \\ 12 & 7 & 8 & 4 & 3 \\ 12 & 7 & 8 & 3 & 4 \end{pmatrix},$$

and for $\bar{G}$ we have

$$\bar{C} = \begin{pmatrix} 8 & 4 & 4 & 0 & 0 \\ 4 & 3 & 2 & 0 & 0 \\ 4 & 2 & 3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

(same labeling of irreducible Brauer characters). Furthermore, for $a(\varphi)$ we get $7, 5, 5, 4, 4$ and for $a(\bar{\varphi})$ we obtain $3, 1, 1, 0, 0$. This shows that $a(\varphi) = 4 + a(\bar{\varphi})$ for all $\varphi \in \mathrm{IBr}_2(G)$.

A more challenging example is the following.

Example 3.3. Let $G = 3^6 : 2M_{12}$ and let $p = 3$. Then G consists only of the principal 3-block and for $a(\varphi)$ where $\varphi \in \mathrm{IBr}_3(G)$ one computes

$$9, 9, 9, 9, 8, 8, 7, 7, 9, 9, 9, 9, 8, 8, 8, 8, 7, 7, 6.$$

The factor group $\bar{G} = 2M_{12}$ has exactly four blocks and we compute for $a(\bar{\varphi})$
block 1: 3,3,3,3,2,2,1,1
block 2: 3,3,3,3,2,2,2,2
block 3: 1,1
block 4: 0
Thus we get $a(\varphi) = a(\bar{\varphi}) + 6$. for all φ.

Many other examples lead to the same connection between the Hilbert divisors of G and $\bar{G} = G/N$ where N is a normal p-subgroup of G.

Conjecture 3.4. If N is a normal p-subgroup of G of order p^n, then $a(\varphi) = a(\bar{\varphi}) + n$ for all $\varphi \in \mathrm{IBr}_p(G)$.

4. Indecomposable quasi-projective Brauer characters

Let C be the Cartan matrix of a block B with $l = l(B) = |\,\mathrm{IBr}_p(B)|$ irreducible Brauer characters. As we have seen at the beginning of Section 2 we know that $\sum_{\varphi \in \mathrm{IBr}_p(B)} a_\varphi \Phi^\circ_\varphi$ with $a = (a_\varphi) \in \mathbb{Z}^l$ is a quasi-projective Brauer character if and only if $Ca \geq 0$. Let ρ denote the map

$$\rho : a \mapsto \sum_{\varphi \in \mathrm{IBr}_p(B)} a_\varphi \Phi^\circ_\varphi.$$

If $cone(C) = \{a \in \mathbb{Z}^l \mid Ca \geq 0\}$ denotes the rational cone defined by C, then its image under ρ describes exactly the set of quasi-projective Brauer characters. According to [14, Theorem 16.4], $cone(C)$ has a unique minimal Hilbert basis which is finite. Its image under ρ is exactly the set of indecomposable quasi-projective Brauer characters which we denote by $\mathrm{H}_p(B)$. Thus we may call $\mathrm{H}_p(B)$ the Hilbert basis of B (with respect to C). According to Theorem 2.1 we have $\{p^{a(\varphi)}\varphi \mid \varphi \in \mathrm{IBr}_p(B)\} \subseteq \mathrm{H}_p(B)$. Computing $\mathrm{H}_p(B)$ may be reduced to compute the minimal Hilbert basis of $cone(C)$ which can be done for not too large parameters by the software system [6].

We start with some examples.

Examples 4.1. a) The principal 2-block of $A_6.2_3$ has two irreducible Brauer characters, say φ and ψ with $a(\varphi) = 4$ and $a(\psi) = 1$. Beneath $2^4\varphi$ and 2ψ there is a third quasi-projective indecomposable character, namely $8\varphi + \psi$. Note that the Cartan matrix of B is given by

$$C = \begin{pmatrix} 16 & 8 \\ 8 & 5 \end{pmatrix}.$$

Hence the splitting of Φ_φ into indecomposable quasi-projectives characters is not unique since

$$\Phi_\varphi = 16\varphi + 4 \times (2\psi) = 2 \times (8\varphi + \psi) + 3 \times (2\psi).$$

b) The principal 2-block B_0 of $PSL(2, 7)$ has three irreducible Brauer characters φ with $a(\varphi) = 3, 3, 3$. The Sylow 2-subgroup is a dihedral group of order 2^3. There are 21 indecomposable quasi-projective Brauer characters.
c) The principal 2-block of $PSL(2, 17)$ has three irreducible Brauer characters with $a(\varphi) = 4, 1, 1$ where $\varphi(1) = 1, 8, 8$. There are exactly 6 indecomposable quasi-projective charcters.
d) The principal 3-block of $PSL(3, 3)$ has 8 irreducible Brauer characters with $a(\varphi) = 3, 2, 2, 2, 2, 3, 2, 2$ where $\varphi(1) = 1, 3, 3, 6, 6, 7, 15, 15$. Note that according to a well-known result of Dipper [7, Section 8.9, Theorem] all vertices coincide with the Sylow 3-subgroup of $PSL(3, 3)$ which is extraspecial of exponent 3. There are exactly 847 indecomposable quasi-projective charcters.

Proposition 4.2. *If B is a p-block of G with Cartan matrix C, then the following are equivalent.*

a) $C = (x_{\varphi\psi} p^{a(\varphi)})$ *where* $x_{\varphi\psi} \in \mathbb{N}_0$.
b) $\det C = \prod_{\varphi \in \mathrm{IBr}_p(B)} p^{a(\varphi)}$.
c) *The Hilbert divisors of* $\mathrm{IBr}_p(B)$ *are exactly the elementary divisors of* C.

Proof. Part a) implies $\det C \geq \prod_{\varphi \in \mathrm{IBr}_p(B)} p^{a(\varphi)}$. On the other hand the proof of Theorem 2.1 a) shows that there is an integer matrix A such that CA is diagonal with entries $p^{a(\varphi)}$ for $\varphi \in \mathrm{IBr}_p(B)$. Thus $\det C \det A = \prod_{\varphi \in \mathrm{IBr}_p(B)} p^{a(\varphi)}$ and we have $\det C \leq \prod_{\varphi \in \mathrm{IBr}_p(B)} p^{a(\varphi)}$. Thus b) is proved.
Suppose that b) holds true. Then $\det A = 1$ and c) follows since the Smith normal form of a matrix is unique.
Finally, part c) implies $\det A = 1$, hence $C = \mathrm{diag}(p^{a(\varphi)} | \varphi \in \mathrm{IBr}_p(B))A^{-1}$ which proves a). □

Theorem 4.3. *Let B be a p-block with Cartan matrix C. Then the following statements are equivalent.*

a) $\mathrm{H}_p(B) = \{p^{a(\varphi)}\varphi \mid \varphi \in \mathrm{IBr}_p(B)\}$.
b) *The Hilbert divisors of* $\mathrm{IBr}_p(B)$ *are the elementary divisors of* C.

Proof. Suppose that part a) holds true. Since C is symmetric, each column of C describes the character of a projective indecompsable module. It is obviously an $\mathbb{N}_0$-linear combination of indecomposable quasi-projective characters, hence, by assumption of the characters $p^{a(\varphi)}\varphi$. Thus C has a shape as in Proposition 4.2 a) and b) follows by Proposition 4.2 c).

Conversely, suppose that b) holds. Let $M = \oplus_{\varphi\in \mathrm{IBr}_p(B)}\mathbb{Z}\varphi$, $N = \oplus_{\varphi\in \mathrm{IBr}_p(B)}\mathbb{Z}\Phi_\varphi$ and $L = \oplus_{\varphi\in \mathrm{IBr}_p(B)} p^{a(\varphi)}\mathbb{Z}\varphi$. Clearly, $|M/N| = \det C$. Since $\det C$ is equal to the product of elementary divisors of C, we get, by the assumption of b),

$$|M/N| = \prod_{\varphi\in \mathrm{IBr}_p(B)} p^{a(\varphi)}.$$

This implies $N = L$ as $L \leq N$ and $|M/L| = \prod_{\varphi\in \mathrm{IBr}_p(B)} p^{a(\varphi)}$. Now, if $\beta \in \mathrm{H}_p(B)$, then $\beta \in L$ and hence $\beta = p^{a(\varphi)}\varphi$ for some $\varphi \in \mathrm{IBr}_p(B)$. □.

Lemma 4.4. *Let B be a p-block of a p-solvable group. Then $\{p^{a(\varphi)}\varphi \mid \varphi \in \mathrm{IBr}_p(B)\} = \mathrm{H}_p(B)$ implies $l(B) = 1$.*

Proof. By [5], we have $c_{\varphi\psi} \leq \min\{|vx(\varphi)|, |vx(\psi)|\}$ for any $\varphi, \psi \in \mathrm{IBr}_p(G)$. According to Proposition 2.8 we know that $p^{a(\varphi)} = |vx(\varphi)|$ and together with Proposition 4.2 we obtain

$$x_{\varphi\psi}|vx(\varphi)| = x_{\varphi\psi}p^{a(\varphi)} = c_{\varphi\psi} \leq \min\{|vx(\varphi)|, |vx(\psi)|\}.$$

This implies $c_{\varphi\varphi} = |vx(\varphi)|$ which contradicts Lemma 2.9 if $|\,\mathrm{IBr}_p(G)| > 1$.

□

Lemma 4.5. *Let B be a p-block with cyclic defect group of order p^d. Then $\{p^{a(\varphi)}\varphi \mid \varphi \in \mathrm{IBr}_p(B)\} = \mathrm{H}_p(B)$ implies $l(B) = 1$.*

Proof. First note that all $\chi \in \mathrm{Irr}(B)$ are of height 0 [4, Ch. VII, Theorem 2.16]. If χ is an end point in the Brauer tree, then $\chi^\circ \in \mathrm{IBr}_p(B)$. Thus, if $l(B) \geq 2$, then there are at least two irreducible p-Brauer characters of height 0. Theorem 2.1 c) implies that $a(\varphi) = d$ for these two Brauer characters φ. This contradicts Theorem 4.3. □

Conjecture 4.6. *Let B be a p-block of any finite group. Then the equality $\{p^{a(\varphi)}\varphi \mid \varphi \in \mathrm{IBr}_p(B)\} = \mathrm{H}_p(B)$ is equivalent to $l(B) = 1$.*

Remarks 4.7. Let B be a p-block of defect d with $l(B) = 2$. Let $\mathrm{IBr}_p(B) = \{\varphi, \psi\}$. By changing the notation of the irreducible Brauer characters if necessary, we may assume that $a(\psi) = d$. If $a(\varphi) = 0$, then $\mathrm{H}_p(B) = \{\varphi, p^d\psi\}$.
One easily sees that the Cartan matrix is given by

$$C = \begin{pmatrix} a & bp^d \\ bp^d & cp^d \end{pmatrix}$$

where $ac - b^2p^d = 1$ and p does not divide a, b, c. Suppose that $b = c = 1$, hence

$$C = \begin{pmatrix} p^d + 1 & p^d \\ p^d & p^d \end{pmatrix}.$$

It follows

$$(*) \qquad \sum_{\chi} d_{\chi\varphi}^2 = c_{\varphi\varphi} = p^d + 1, \sum_{\chi} d_{\chi\psi}^2 = c_{\psi\psi} = p^d, \sum_{\chi} d_{\chi\varphi} d_{\chi\psi} = c_{\varphi\psi} = p^d.$$

Thus

$$\sum_{\chi} (d_{\chi\varphi} - d_{\chi\psi})^2 = c_{\varphi\varphi} - 2c_{\varphi\psi} + c_{\psi\psi} = p^d + 1 - 2p^d + p^d = 1.$$

This shows that there exists χ_0 such that $|d_{\chi_0\varphi} - d_{\chi_0\psi}| = 1$ and $d_{\chi\varphi} = d_{\chi\psi}$ for $\chi \neq \chi_0$. Now

$$d_{\chi_0\varphi}^2 = (d_{\chi_0,\psi} \pm 1)^2 = d_{\chi_0,\psi}^2 \pm 2d_{\chi_0\psi} + 1$$

together with the conditions in (*) force $d_{\chi_0\psi} = 0$. Thus

$$|G|_p \mid \varphi(1) \mid \chi_0(1)$$

and χ_0 is of defect 0, a contradiction.

5. A characterization of p-nilpotent groups

All quasi-projective ordinary characters in a block B are projective if and only if all irreducible Brauer characters in B are quasi-liftable [15, Theorem 1.4]. The property that all quasi-projective Brauer characters in B are projective turned out to have an easy answer.

Proposition 5.1. *If $p^{a(\varphi)}\varphi$ is projective for some $\varphi \in \mathrm{IBr}_p(B)$, then $l(B) = 1$.*

Proof. For $\varphi \in \mathrm{IBr}_p(B)$, we have

$$p^{a(\varphi)}\varphi \sum_{\beta \in \mathrm{IBr}_p(B)} a_\beta \Phi_\beta^\circ$$

with $a_\beta \in \mathbb{N}_0$. In particular, $\Phi_\beta^\circ = p^n\varphi$ for some $\beta \in \mathrm{IBr}_p(B)$ and $0 \leq n \leq a(\varphi)$. Clearly $\beta = \varphi$. Since the projective characters Φ_γ in a block are connected [8, Ch. VII, Theorem 12.4] we get $\mathrm{IBr}_p(B) = \{\varphi\}$. □

Corollary 5.2. *All quasi-projective Brauer characters in the principal p-block of G are projective if and only if G is p-nilpotent.*

Proof. This follows immediately by Proposition 5.1 and [8, Ch. VII, Theorem 14.9]. □

6. Further questions and remarks

Question 6.1. Let B be a p-block of G of defect d and let $\varphi \in \mathrm{IBr}_p(B)$. Is it true that $a(\varphi) = d$ always implies that $\mathrm{ht}(\varphi) = 0$?

Clearly the above is true for p-solvable groups and blocks with cyclic defect groups. Note that very often (but not always) $a(\varphi) + \mathrm{ht}(\varphi) = d$. A counterexample is the group McL for $p = 2$.

Question 6.2. What does it mean for a block B if $p^{a(\varphi)} = |vx(\varphi)|$ for all $\varphi \in \mathrm{IBr}_p(B)$?

Remark 6.3. Let $\Phi = \sum_{\varphi \in \mathrm{IBr}_p(G)} a_\varphi \varphi$ with $p \mid a_\varphi \in \mathbb{N}_0$. Suppose that Φ is the Brauer character of a projective module. Then $\tilde{\Phi} = \sum_{\varphi \in \mathrm{IBr}_p(G)} \frac{a_\varphi}{p} \varphi$ is in general not quasi-projective. As an example we may take the alternating group $G = A_5$ on 5 letters, $p = 2$ and for Φ the character Φ_1 of the principal indecomposable module. The coefficients of the indecomposable quasi-projective Brauer characters of the principal 2-block (w.r.t. $\mathrm{IBr}_2(B_0)$) are given by $(4,0,0), (0,2,0), (0,0,2), (2,1,0), (2,0,1)$ and $(0,1,1)$. Note that Φ_1 has coefficients $4, 2, 2$.

Question 6.4. Let $p^{d_1} \leq \cdots \leq p^{d_{l-1}} < p^{d_l} = p^d$ denote the elementary divisors of the Cartan matrix of a block B. According to Theorem 2.1 d) we know that $\sum_{i=1}^{l} d_i \leq \sum_{\varphi \in \mathrm{IBr}_p(B)} a(\varphi)$. Is it possible to label the irreducible Brauer characters of B as $\varphi_1, \ldots, \varphi_l$ such that

$$d_i \leq a(\varphi_i) \quad \text{for all } i = 1, \ldots, l?$$

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